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A note on generalization of Zermelo navigation problem on Riemannian manifolds with strong perturbation

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Abstract

We generalize the Zermelo navigation on Riemannian manifolds (M,h), admitting a space dependence of a ship's speed $0 < |u(x)|_h \leq 1$ in the presence of a perturbation $\tilde{W}$ determined by a strong (critical) velocity vector field satisfying $|\tilde{W}(x)|_h = |u(x)|_h$, with application of Finsler metric of Kropina type.

1 Introduction

The objective in the navigation problem of Zermelo is to find the minimum time paths of a ship sailing on a sea M, with the presence of a wind determined by a vector field W. The problem was formalized and investigated by E. Zermelo (1931) in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, cf. [24, 25], and generalized considerably (2004) in [5] for the case when sea is a Riemannian manifold (M,h), however under the assumption that a wind W is a time-independent weak wind, i.e. $h(W,W) < 1$. In the absence of a perturbation the solutions to the problem are simply h-geodesics of M. Note that the original solution given by E. Zermelo admitted a strong wind and time dependence. Thus, in a special case the problem may be treated as purely geometric. It has been found out that the trajectories which minimize travel time are exactly the geodesics of a special Finsler type F, that is Randers metric. Briefly, the

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solutions to the problem are the flows of Randers geodesics. The condition on strong convexity, i.e. $|W|_h < 1$ ensures then that F is a positive definite Finsler metric. Furthermore, there is an equivalence between Randers metrics and Zermelo's problems [5, 11].

In [23] the authors showed that Zermelo's navigation problem has another solution in Finsler geometry in the case when the wind becomes stronger. Also, such a problem in a complex setting, that is, on Hermitian manifolds, was considered in [1]. This means that there is a wind acting of the same impact as a maximal power of ship's engine. Precisely, it was assumed that $h(W, W) = 1$ (critical perturbation). Note that the problem was considered in the original formulation when a ship sails with Riemannian unit speed, i.e. $h(u, u) = 1 = const$. Obviously, since the ship's speed $|u|_h$ and the wind force $|W|_h$ are equal, unlike the Randers case, the ship cannot proceed anymore against the wind. So, following the direction $u = -W$ implies that the resulting velocity v vanishes. Geometrically, in each tangent space T_xM the unit sphere of the new metric F is the W-translate of the Riemannian h-unit sphere. However, differently from the Randers case, the former passes through the origin of T_xM and thus F cannot be a Finlser metric in the classic sense [23].

Setting as a reference point Zermelo's formulation of the problem we may ask whether a ship must proceed at a constant maximum speed relative to the surrounding Riemannian sea, i.e. $|u|_h = 1$. This assumption we have already dropped considering the problem on Riemannian manifolds, however being in the case of a background weak wind which guarantees a full control of navigating ship (cf. [15]). Recently we have also investigated the analogous problems on Hermitian manifolds in complex Finsler geometry (cf. [2, 3]). Reviewing the bibliography in this scope one may find the paper in the calculus of variations by A. de Mira Fernandes [12] who accomodated shortly after Zermelo's contribution a varying magnitude ship's velocity. Having added the extra degree of freedom the author allowed a time and space dependent velocity and solved the corresponding problem with the Euclidean background, namely in $\mathbb{R}^n$. Therefore, he has generalized the results of E. Zermelo [24, 25] and T. Levi-Civita [17] for the Euclidean spaces to the case where the air speed of a plane is a preassigned function of position and time. Also, the subsequent equations for the flight path of least time obtained by K. Arrow [4], who considered a passage with $\mathbb{S}^2$-background, implied earlier results achieved by T. Levi-Civita. De Mira Fernandes showed that the change in $|u|_h$ with time has no effect on the formula for the shortest time passage (time-optimal ship's heading) while that with space has the same effect as the corresponding change in wind [4].

The above contribution was also referred in the modern approach (cf. [13]) in a discussion how, when both W and $|u|_h$ are space but not time dependent, it can be recast in a purely geometric form as geodesics of a Randers geometry or as null geodesics in a stationary space-time. Let us note that the Zermelo navigation as a method plays an active and crucial role in modern physics, in particular in quantum mechanics. In this regards, see, for instance, the expositions in [9, 22, 21, 8, 6, 7, 14]. Furthermore, the concept admitting a varying magnitude ship's velocity gives rise to optimize our recent applied study of the search models based on time-minimal paths in real navigational applications (cf. [14, 16]). In the current investigation on Riemannian manifolds for the case of a strong wind we are also going to drop the standard assumption on a constant unit speed. We aim to present our glance at the problem with different starting point and therefore contribute to the previous findings (cf. [23]) by introducing a space dependence of a ship's speed, $0 < |u(x)|_h \leq 1$.

2 Glance at previous findings from a different perspective

Let a pair (M, h) be a Riemannian manifold where $h = h_{ij}dx^i \otimes dx^j$ is a Riemannian metric and the corresponding norm-squared of tangent vectors $y \in T_xM$ is denoted by $|y|_h^2 = h_{ij}y^iy^j = h(y, y)$. In contrast to [23] we begin with a Riemannian manifold (M, h) and a vector field $\tilde{W} = \tilde{W}^i \frac{\partial}{\partial x}$ on M which need not be of h-unit length. We admit that both ship's speed $|u(x)|_h$ and wind W are space-dependent with $0 < |u(x)|_h = |\tilde{W}(x)|_h \leq 1$. Such stronger perturbation will be called critical. Thus, a ship makes a way unceasingly through the water, but not necessarily over ground. We compute the new Finsler metric $\tilde{F}$ similarly as treated in [23], with the refinement of the initial indicatrix-based equation. To reduce the clutter we also adopt the same notations if not otherwise stipulated. We aim to obtain the metric $\tilde{F}$ as the solution to the refined equation including the new variable. We have

$$\left| \frac{y}{\tilde{F}(x, y)} - \tilde{W} \right| = |u(x)|. \tag{1}$$

It thus follows from the definition of the inner product

$$h_{ij}(y^i - \tilde{F}\tilde{W}^i)(y^j - \tilde{F}\tilde{W}^j) = |u|^2\tilde{F}^2. \tag{2}$$

Hence,

$$(|u|^2 - |\tilde{W}|^2)\tilde{F}^2 + 2h(y, \tilde{W})\tilde{F} - |y|^2 = 0. \tag{3}$$

By assumption on the equality of the norms we are thus led to

$$\tilde{F}(x, y) = \frac{|y|_h^2}{2h(y, \tilde{W}(x))}. \tag{4}$$

From the above concerned assumption it is implied that $\tilde{W} \neq 0$; $y \neq 0$. We obtained the metric of the analogous form as F in the standard case when $h(\tilde{W}, \tilde{W}) = 1$. We also require that on M there must exist a vector field $\tilde{W}$ without zeros. Therefore, having in mind the topological restrictions coming from the Poincaré-Hopf theorem one restricts the structures (M, h) which the theory under consideration can be applied to. In particular, we exlude $\mathbb{S}^2$ since it follows that for any compact regular 2-dimensional manifold with non-zero Euler characteristic any continuous tangent vector field has at least one zero.

Remark 2.1. Under a strong (critical) perturbation $|\tilde{W}|_h = 1$ formula (4) as a special case leads to the metric F according to [23] in the standard formulation of the Zermelo navigation problem on Riemannian manifolds, i.e. with $h(u, u) = 1$.

Let us observe that

$$\tilde{F}(x,y) = \frac{|y|_h^2}{2h(y, |u(x)|_h W(x))} = \frac{1}{|u(x)|_h} F(x,y) \tag{5}$$

where $F(x,y) = \frac{|y|_h^2}{2h(y,W(x))}$ in the standard expression. From (5) it implies the following

Corollary 2.1. *$\tilde{F}$ is a Finsler metric conformal to F.*

This recalls the scenario in the generalized Randers case in the absence of a wind. Then, however, the resulting Randers metric is Riemannian and conformal to the corresponding background Riemannian metric h; see Proposition 2.5 in [15]. The adequate and wider investigation on conformal and weakly conformal Finsler geometry can be found, in particular, in [20, 19]. To proceed, we can simply assume that

$$\tilde{a}_{ij}(x) = h_{ij}, \qquad \tilde{b}_i(x) = 2\tilde{W}_i. \tag{6}$$

Hence,

$$\tilde{b}^2 = \tilde{a}^{ij}\tilde{b}_i\tilde{b}_j = 4|u(x)|_h^2 \tag{7}$$

while in the original setting one would then get $b^2 = a^{ij}b_ib_j = 4 = const.$ In order to avoid a constant function here as the obtained metrics could be a subject to such a constraint, a conformal factor $e^{-k(x)}$ has been applied making use of some smooth function $k(x)$ on M. For comparison, to be in line with [23], if we use an analogous conformal factor $e^{-\tilde{k}(x)}$, where $\tilde{k}(x)$ is also some smooth function on M, then we get $\tilde{F}(x,y) = \frac{h(y,y)}{2h_{ij}\tilde{W}^j y^i} = \frac{e^{-\tilde{k}(x)}h_{ij}y^iy^j}{2e^{-\tilde{k}(x)}\tilde{W}_i y^i}$. Therefore, taking

$$\tilde{a}_{ij}(x) = e^{-\tilde{k}(x)}h_{ij}, \qquad \tilde{b}_i(x) = 2e^{-\tilde{k}(x)}\tilde{W}_i, \tag{8}$$

yields the special Finsler type metric, namely the Kropina metric

$$\tilde{F}(x,y) = \frac{\tilde{a}_{ij}(x)y^i y^j}{\tilde{b}_i(x)y^i} = \frac{\tilde{\alpha}^2(x,y)}{\tilde{\beta}(x,y)}. \tag{9}$$

$\tilde{F}(x,y)$ is composed of the new Riemannian metric $\tilde{\alpha} = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ and a 1-form $\tilde{\beta} = \tilde{b}_i(x)y^i$ where $\tilde{b}^2 = 4|u(x)|_h^2 e^{-\tilde{k}(x)}$. Conversely, if we put $h_{ij} = e^{\tilde{k}(x)}\tilde{a}_{ij}$ and $\tilde{W}_i(x) = \frac{e^{\tilde{k}(x)}\tilde{b}_i}{2}$, where

$$\tilde{k}(x) = 2\ln\left(\frac{2|u(x)|_h}{\tilde{b}(x)}\right) \tag{10}$$

then we obtain the initial navigation data in terms h and $\tilde{W}$ which solution to the problem is exactly the Kropina metric (9). To compare, recall $k(x) = \ln\frac{4}{b^2(x)}$ in the original setting. Note that it is sufficient to apply (6) in order to obtain the same form of the Kropina metric given by (9). Therefore, $\tilde{b}^2 \neq$ *const.* with $h(u,u) \neq$ *const.* Fulfilling the definition of Finsler metric which is positive definite, the function (9) is not defined on all TM, but only on a domain $\{(x,y) \in TM : \tilde{\beta} > 0\}$. Therefore, we exclude the case when $u = -\tilde{W}$. Following [23] we have

Definition 2.2. Let (M,h) be an n-dimensional Riemannian space, $\tilde{W}$ a vector field globally defined on M. Let $\tilde{a}_{ij}$ and $\tilde{b}_i$ be given by (8) and denote the Kropina metric by $\tilde{F}$, where $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$. Then $\tilde{F}$ will be called $\tilde{U}$-Kropina metric.

Recall that since the Kropina metrics defined globally on M are considered, the above mentioned topological restrictions to their existence occur. For more details see Propositions 5.2 and 5.13 in [23]. One sees immediately that in the case of $h(\tilde{W},\tilde{W}) = 1 =$ *const.* $\tilde{U}$-Kropina metric becomes U-Kropina metric defined in the standard presentation, that is Kropina metric with unit vector field. Recalling (6) and (9) it results that the generalization preserves the original Riemannian metric α but changes the 1-form β. Comparing the resulting Finsler metrics we observe that $\tilde{\alpha} = \alpha$ and $\tilde{\beta} \neq \beta$ since $\tilde{W}_i \neq W_i$ for $h(u,u) \neq 1$. The difference is made by perturbing wind what, in other words, is connected to the fact of admitting a ship's speed $|u|_h$ to vary in space. Let us summarize after the refinement of the previous investigation which became the point of reference and the motivation for our study. We thus obtain the following

Proposition 2.2. *A metric $\tilde{F}$ is of $\tilde{U}$-Kropina type if and only if it solves the generalized Zermelo navigation problem on a Riemannian manifold (M,h),* *with varying in space ship's speed* $0 < |u(x)|_h \leq 1$ *in the presence of a strong (critical) wind* $\tilde{W}(x)$ *which satisfies* $|\tilde{W}|_h = |u|_h$.

Remark that we exclude here $|W|_h = 0$ unlike the generalized Randers case with a spatial function $|u(x)|_h$ in the presence of a weak perturbation, say W_R, i.e. $0 \leq h(W_R, W_R) < h(u, u)$, where the solutions to the problem are then determined by the Riemannian metric conformal to h. From (5) it yields that the resulting Kropina geodesics of F and $\tilde{F}$ with $|u|_h = const.$ trace the same curves, however the corresponding speeds differ and therefore the times of travel between given points change. This refers to the particular situation in the generalized Randers case, i.e. with $W_R = 0$ and $h(u, u) = const.$ Then, however, Randers metric is reduced to the corrresponding background Riemannian metric h up to scaling; for more details see the study in [15]. Such case also corresponds to a pair of conformal homothetic Finsler metrics, that is a special case of weakly conformally equivalent Finsler metrics considered in [20]. Going further, a glance at the new metric (4) and (5) leads to

Corollary 2.3. *With arbitrary generalized navigation data* $(h, |u(x)|_h, \tilde{W}(x))$ *a transit time of existing, nonzero ($u \neq -\tilde{W}$) solution to the generalized Zermelo navigation problem on Riemannian manifolds in the presence of a strong (critical) wind satisfying* $|\tilde{W}|_h = |u|_h \neq 1$ *is greater than a transit time of the corresponding solution to the standard Zermelo navigation problem.*

Proof. For any piecewise C^∞ curve ℓ in M, the $\tilde{F}$- length of ℓ denoted by $\mathcal{L}_{\tilde{F}}(\ell)$ is equal to the time for which the object travels along it, i.e. $T = \int_0^T \tilde{F}(\dot{\ell}(t))dt = \mathcal{L}_{\tilde{F}}(\ell)$. Let γ, $\tilde{\gamma}$ be F- and $\tilde{F}$-geodesic, respectively, where $\tilde{F}$ is given by (4). For any nonzero $y \in T_xM$ $F(y), \tilde{F}(y) > 0$. The function $|u(x)|_h$ is variable in space or constant with $0 < h(u, u) \leq 1$. Since $u \neq -\tilde{W}$ the resultant $v > 0$. The equality of the lengths $\mathcal{L}_{\tilde{F}}$ and $\mathcal{L}_F$ holds if and only if $|u|_h = 1 = const.$, then $\tilde{F}(y) := F(y)$. Otherwise, by (5) $\tilde{F}(y) > F(y)$ for any scenario obtained from the triples $(h, \tilde{W}(x), |u(x)|_h)$, thus for any spatial function $|u(x)|_h$ or, equivalently, $|\tilde{W}(x)|_h$, where $|\tilde{W}|_h = |u|_h$. Note that $\mathcal{L}_{\tilde{F}}(\tilde{\gamma}) \geq \mathcal{L}_F(\tilde{\gamma})$ and $\mathcal{L}_F(\tilde{\gamma}) \geq \mathcal{L}_F(\gamma)$ as the geodesic minimizes the length. From transitivity we are thus led to the inequality $\mathcal{L}_{\tilde{F}}(\tilde{\gamma}) \geq \mathcal{L}_F(\gamma)$. $\square$

Remark that the presence of $|u(x)|_h$ in the above expression of navigation data may actually be inessential. If perturbing vector field is a priori fixed then it can be removed, since given $\tilde{W}$ determines $|u|_h$ by $|\tilde{W}|_h$. Nevertheless, we let it to emphasise its new role in the considered approach to the problem inasmuch as we admit $|u(x)|_h$ to be set initially, without being determined by $\tilde{W}$. Next, let us observe that unlike the Randers case, where the entire space

(M, h) can be covered with the time-minimal paths, not all the positions $x \in M$ are now available for navigating ship any more since a wind is of stronger force. Therefore, one needs to consider the existence of solutions to posed Zermelo problems. Also, from the above it yields the following, somewhat contrariwise formulated, corollary.

Corollary 2.4. *Let a space-dependent ship's speed $|u(x)|_h$ vary along the existing solution to the generalized Zermelo navigation problem on a Riemannian manifold (M, h) in the presence of a strong (critical) perturbation $\tilde{W} \neq -u$ with $|u|_h = |\tilde{W}|_h$. Then the stronger perturbation acts on a ship the shorter travel time is.*

It implies that a ship reaches her destination in the absolutely shortest time when the strongest perturbation blows, what one may find self-contradictory. Indeed, if $h(\tilde{W}, \tilde{W}) = 1$ the passage is time-minimal amidst all the possible combinations of generalized navigation data. Namely, the original formulation of the problem brings optimal solution in comparison to the other ones modified by the new variable $|u(x)|_h$. Obviously, increasing a wind's force causes that a ship's speed is also boosted to hold the constant ratio $\frac{|\tilde{W}|_h}{|u|_h} = 1$ as assumed in the problem under consideration.

Now, let us take a look at different straightforward scenario in the presence of the same strong and varying in magnitude wind $\tilde{W}(x)$, with two Riemannian seas (M, h) and $(M, \hat{h})$ which are determined by the conformal background Riemannian metrics $\hat{h}$ and h, where $\hat{h} = \frac{1}{|u|_h^2} h$, $h = (h_{ij}), \hat{h} = (\hat{h}_{ij})$. Thus, $\hat{h}(\tilde{W}, \tilde{W}) = 1 = \hat{h}(u, u)$ since, by assumption, $h(\tilde{W}, \tilde{W}) = h(u, u)$. Therefore, this gives

$$\hat{F}(x, y) = \frac{\hat{h}(y, y)}{2\hat{h}(y, \tilde{W}(x))} = \frac{h(y, y)}{2|u(x)|_h^2 \hat{h}(y, \tilde{W}(x))} = \frac{|y|_h^2}{2h(y, \tilde{W}(x))}. \tag{11}$$

Hence, recalling (4) it yields the equality of the above Kropina metrics, namely $\hat{F} = \tilde{F}$. We are thus led to

Lemma 2.5. *The time-minimal paths of the conformal background Riemannian metrics $\hat{h}$ and h, where $\hat{h} = |u(x)|_h^{-2} h$, perturbed by a strong (critical) varying in space wind $\tilde{W}(x)$ which satisfies $|\tilde{W}|_h = |u|_h$, are represented by the same Kropina geodesics.*

The analogous fact we also showed, in particular, in the complex setting providing deeper analysis for the complex Randers and Kropina cases on Hermitian manifolds (cf. [2, 3]). In the sequel, we present the flow of Kropina geodesics in the generalized approach to the Zermelo navigation problem, with the presence of a strong wind including the influence of a spatial function $|u(x)|_h$. For clarity we also compare it to the corresponding solution obtained from the original expression of the problem on the same Riemannian sea (M, h).

3 Example

With the topological restrictions in mind which refer to the existence of globally defined Kropina metrics on M, admitting however $\mathbb{S}^{2m-1}$ or $\mathbb{E}^n$, in what follows we present the example with the Euclidean background, namely $\mathbb{E}^2$. Considering dimension two we denote the position coordinates (x^1, x^2) by (x, y) and expand arbitrary tangent vectors $y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2}$ at (x^1, x^2) as $(x, y; u, v)$ or $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$. We also express a ship's speed $|u|_h$ as $|U|$ and the resulting speed $|v|_h$ as $|V|$. Thus, from (4) we obtain

$$\tilde{F}(x, y; u, v) = \frac{h_{11}u^2 + 2h_{12}uv + h_{22}v^2}{2(h_{11}u\tilde{W}^1 + h_{12}u\tilde{W}^2 + h_{21}v\tilde{W}^1 + h_{22}v\tilde{W}^2)}. \tag{12}$$

After having set $M := \mathbb{R}^2$ we get

$$\tilde{F}(x, y; u, v) = \frac{u^2 + v^2}{2(u\tilde{W}^1 + v\tilde{W}^2)} \tag{13}$$

where simply $|(u, v)|_h = \sqrt{u^2 + v^2}$. Without loss of generality let us consider the strong unit wind W represented by

$$W(x, y) = \cos(x + y)\frac{\partial}{\partial x} + \sin(x + y)\frac{\partial}{\partial y}. \tag{14}$$

Hence,
$|W(x, y)|_h = \sqrt{(W^1(x, y))^2 + (W^2(x, y))^2} = \sqrt{\cos^2(x + y) + \sin^2(x + y)} = 1$ $\forall\ (x, y) \in \mathbb{R}^2$. Consequently, for the applied perturbation (14) the form of the resulting metric in the original expression yields

$$F(x, y; u, v) = \frac{u^2 + v^2}{2(uW^1 + vW^2)} = \frac{u^2 + v^2}{2[u\cos(x + y) + v\sin(x + y)]}. \tag{15}$$

Let $|u(x)|_h$ be smooth and positive determined by a Gaussian function which is expressed in the general form $f(x) = \bar{a}e^{-\frac{(x-\bar{b})^2}{2\bar{c}^2}}$, where $\bar{a}, \bar{b}, \bar{c}$ are the real constants. For instance, let $|U(x, y)| = \frac{2}{3}\exp\left(-\frac{y^2\sin^2(x+y)}{\pi}\right) + \frac{1}{3}$. Thus,

$\forall\ (x,y) \in \mathbb{R}^2 \quad |U(x,y)| \in \left(\frac{1}{3},1\right] \subset (0,1]$ as generally required. The new non-unit wind $\tilde{W}$ blowing on the Euclidean sea yields

$$\tilde{W}(x,y) = \frac{1}{3}\left(2e^{-\frac{1}{\pi}y^2\sin^2(x+y)}+1\right)\cos(x+y)\frac{\partial}{\partial x} + \frac{1}{3}\left(2e^{-\frac{1}{\pi}y^2\sin^2(x+y)}+1\right)\sin(x+y)\frac{\partial}{\partial y}. \quad (16)$$

Let us note that $|W(x,y)| = |U(x,y)|^{-1}|\tilde{W}(x,y)|$. The contour plot and the stream density plot, taking the scalar field to be the norm of the perturbation $\tilde{W}$, are presented in Figure 1.

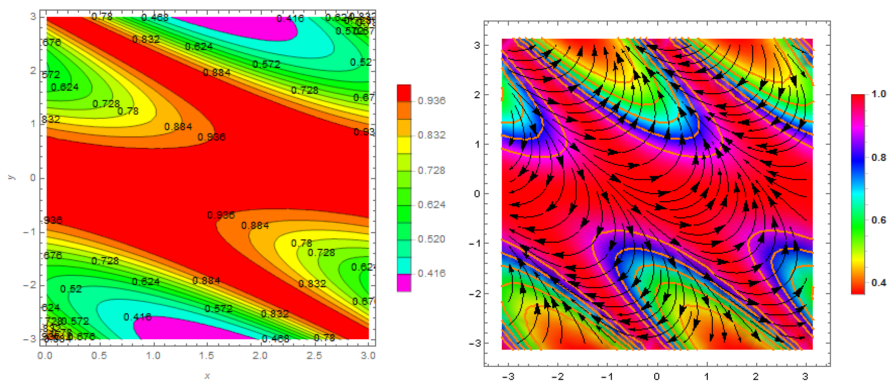

Figure 1: The contour plot (on the left) and the stream density plot (on the right) of the perturbation $\tilde{W}$ given by (16).

By assumption, $|\tilde{W}(x,y)|_h = \frac{2}{3}\exp\left[-\frac{1}{\pi}y^2\sin^2(x+y)\right]+\frac{1}{3}$. For example, with $\varphi_0 = 0$ a ship commences the voyage starting from the origin with a wind, i.e. $U = \tilde{W}$ and $(\dot{x},\dot{y}) = (2,0)$ at the maximum resulting speed which equals $|V| := |U| + |\tilde{W}| = 2$ since then $V := 2U$, where $\varphi = \varphi(t)$ is the angle measured counterclockwise which the vector of the relative velocity U forms with x-axis. Recall that the function (9) is not defined on all TM, but only on a domain $\{(x,y;u,v) \in TM : \tilde{\beta} > 0\}$. Thus, we have $\varphi_0 \in [0,2\pi) \setminus \{\pi\}$. Clearly, with $\varphi_0 = \pi$, where $\tilde{W}(0,0) = (1,0)$, one gets $(\dot{x},\dot{y}) = (0,0)$. This means that though a ship proceeds ceaselessly through the water ($|U| > 0$), it is stopped over ground, i.e. the resulting speed $|V| = 0$. Such a scenario does not occur in the case of Randers metric including the generalized version of the problem in the presence of a weak wind where $|W_R|_h < |u|_h$, cf. [15]. For the perturbation (16) we obtain the form of the resulting metric as follows

$$\tilde{F}(x,y;u,v)=\frac{3\left(u^2+v^2\right)\exp\left(\frac{1}{\pi}y^2\sin^2(x+y)\right)}{2\left[\exp\left(\frac{1}{\pi}y^2\sin^2(x+y)\right)+2\right]\left[u\cos(x+y)+v\sin(x+y)\right]}. \tag{17}$$

After having computed the spray coefficients according to [11], we obtain F-geodesic equations for the Kropina metric (15) in the standard setting. We write $\delta=x+y$ for short and the result is

$$\begin{cases} \ddot{x}+\frac{1}{2(\dot{x}^2+\dot{y}^2)}\left[\left(4\dot{x}^3\dot{y}+4\dot{x}^2\dot{y}^2-\dot{x}^4+\dot{y}^4\right)\sin^2\delta+\frac{1}{2}\left(\dot{x}^4+\dot{y}^4\right)\sin 2\delta\right. \\ \left.+\dot{x}\dot{y}\left(-3\dot{x}\dot{y}+2\dot{x}^2-2\dot{y}^2\right)\sin 2\delta+2\dot{y}^2\left(2\dot{x}\dot{y}-\dot{x}^2+\dot{y}^2\right)\cos^2\delta\right]=0 \\ \\ \ddot{y}-\frac{1}{2(\dot{x}^2+\dot{y}^2)}\left[\dot{x}\left(2\dot{x}\left(2\dot{x}\dot{y}+\dot{x}^2-\dot{y}^2\right)\sin^2\delta+\dot{y}\left(-3\dot{x}\dot{y}-2\dot{x}^2+2\dot{y}^2\right)\sin 2\delta\right)\right. \\ \left.+\left(4\dot{x}^2\dot{y}^2+4\dot{x}\dot{y}^3+\dot{x}^4-\dot{y}^4\right)\cos^2\delta+\frac{1}{2}\left(\dot{x}^4+\dot{y}^4\right)\sin 2\delta\right]=0 \end{cases} \tag{18}$$

In the presented generalization the influence of a spatial function $|U|$ or, equivalently, $|\tilde{W}|$ is noticable in the system of $\tilde{F}$-geodesic equations which correspond to the Kropina metric (17). Abbreviating $\omega=\exp\left(\frac{1}{\pi}y^2\sin^2\delta\right)+2$, this gives

$$\begin{cases} \ddot{x}+\frac{1}{2\pi\omega(\dot{x}^2+\dot{y}^2)}\left\{16y\dot{x}\dot{y}^3\sin^4\delta-\pi\left(-4\dot{x}^3\dot{y}-4\dot{x}^2\dot{y}^2+\dot{x}^4-\dot{y}^4\right)\omega\sin^2\delta\right. \\ +\sin 2\delta\left[\pi\left(\left(2\dot{x}^3\dot{y}-3\dot{x}^2\dot{y}^2-2\dot{x}\dot{y}^3\right)\omega+\dot{x}^4+\dot{y}^4\right)\right. \\ \left.-y\left(-4(y+1)\dot{x}^3\dot{y}-6y\dot{x}^2\dot{y}^2+4y\dot{x}\dot{y}^3+y\dot{x}^4+y\dot{y}^4\right)\sin 2\delta\right] \\ +8y^2\dot{x}^2\left(2\dot{x}\dot{y}+\dot{x}^2-\dot{y}^2\right)\sin\delta\cos^3\delta+2\pi\dot{y}^2\left(2\dot{x}\dot{y}-\dot{x}^2+\dot{y}^2\right)\omega\cos^2\delta \\ +\frac{1}{2}\sin 2\delta\left[4y\left(2(2y+3)\dot{x}^2\dot{y}^2+4y\dot{x}\dot{y}^3+(y-1)\dot{x}^4-(y+1)\dot{y}^4\right)\sin^2\delta\right. \\ \left.\left.+\pi\left(\dot{x}^4+\dot{y}^4\right)(\omega-2)\right]\right\}=0 \\ \\ \ddot{y}-\frac{1}{2\pi(\dot{x}^2+\dot{y}^2)\omega}\left\{-8y\dot{y}^2\left(\dot{y}^2-\dot{x}^2\right)\sin^4\delta+2\pi\dot{x}^2\left(2\dot{x}\dot{y}+\dot{x}^2-\dot{y}^2\right)\omega\sin^2\delta\right. \\ +\sin 2\delta\left[\pi\left(\left(-2\dot{x}^3\dot{y}-3\dot{x}^2\dot{y}^2+2\dot{x}\dot{y}^3\right)\omega+\dot{x}^4+\dot{y}^4\right)\right. \\ \left.+y\left(4y\dot{x}^3\dot{y}-2(3y+2)\dot{x}^2\dot{y}^2-4y\dot{x}\dot{y}^3+(y+1)\dot{x}^4+(y-1)\dot{y}^4\right)\sin 2\delta\right] \\ +4y^2\left(-4\dot{x}^3\dot{y}-4\dot{x}^2\dot{y}^2+\dot{x}^4-\dot{y}^4\right)\sin\delta\cos^3\delta \\ +\pi\left(4\dot{x}^2\dot{y}^2+4\dot{x}\dot{y}^3+\dot{x}^4-\dot{y}^4\right)\omega\cos^2\delta+\frac{1}{2}\sin 2\delta\left[\pi\left(\dot{x}^4+\dot{y}^4\right)(\omega-2)\right. \\ \left.\left.-8y\dot{y}\left(-y\dot{x}^2\dot{y}+2(y+1)\dot{x}\dot{y}^2-2\dot{x}^3+y\dot{y}^3\right)\sin^2\delta\right]\right\}=0. \end{cases} \tag{19}$$

The form of the initial conditions including the optimal control $\varphi(t)$ under perturbing vector field reads $x(0)=x_0\in\mathbb{R}$, $y(0)=y_0\in\mathbb{R}$, and for the first derivative (a tangent vector)

$$\begin{aligned}\dot{x}(0)&=\tilde{W}^1(x_0,y_0)+|U(x_0,y_0)|\cos\varphi_0 \\ &=|U(x_0,y_0)|(\cos(x_0+y_0)+\cos\varphi_0):=1+\cos\varphi_0,\end{aligned} \tag{20}$$

$$\dot{y}(0) = \tilde{W}^2(x_0, y_0) + |U(x_0, y_0)| \sin\varphi_0 \\ = |U(x_0, y_0)|(\sin(x_0 + y_0) + \sin\varphi_0) := \sin\varphi_0. \quad (21)$$

The last relations can be derived by direct consideration of the planar equations of motion including the representation of the vector components of ship's velocity and the new background wind. When the families of the time-minimal paths coming from the same fixed point $x \in M$ are considered, φ_0 plays the role of the parameter which rotates the tangent vector of unperturbed Riemannian geodesic. To provide some numerical computations and to generate the graphs we use Mathematica 10.4 from Wolfram Research. The time-efficient paths in both scenarios, that is the Kropina F- (black) and $\tilde{F}$-geodesics (red) starting from the origin, with the corresponding strong background winds are presented in Figure 2. We set the increments $\Delta\varphi_0 = \frac{\pi}{8}$ and $t = 10$. The obtained solutions are also confronted accordingly in Figure 3. The graphical

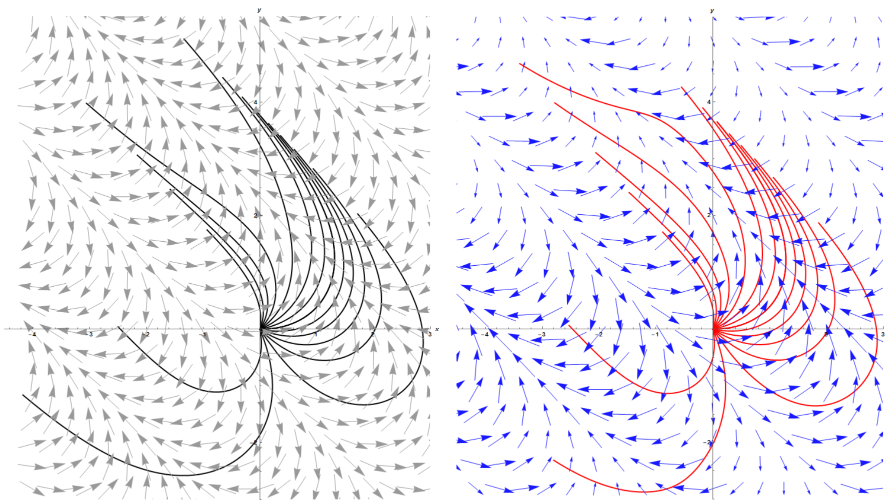

Figure 2: The Kropina F-geodesics (black) with the unit background strong (critical) wind W (grey) and $\tilde{F}$-geodesics (red) with the new non-unit background strong (critical) wind $\tilde{W}$ (blue), with the increments $\Delta\varphi_0 = \frac{\pi}{8}$; $t = 10$.

interpretation of Corollary 2.3 with reference to the example is presented in Figure 4, where three pairs of F- (black) and $\tilde{F}$- isochrones (red) are compared. It implies that for the corresponding times t the former includes the latter what is the consequence of the influence of applied space-dependent ship's speed. The isochrone of $\tilde{F}$ is similar to the isochrone of F with similarity ratio $|u(x)|_h := |U(x, y)|$.

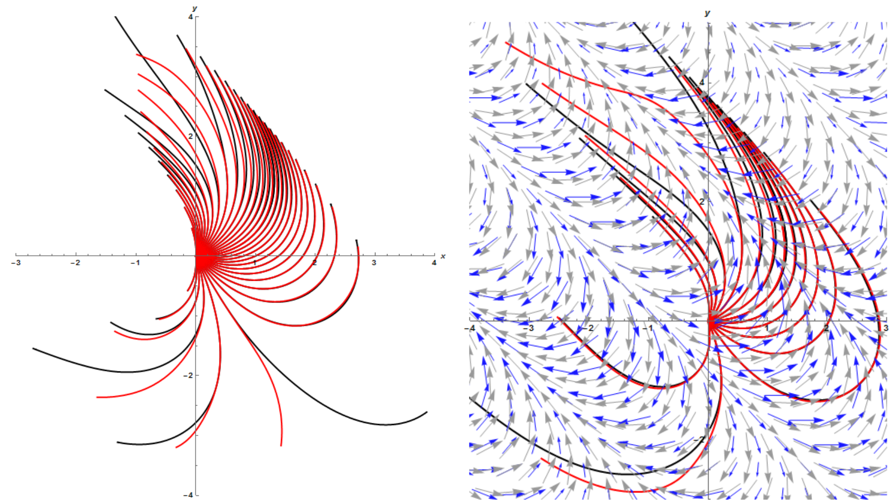

Figure 3: The Kropina $\tilde{F}$-geodesics (red) starting from the origin compared to the Kropina F-geodesics (black) with the increments $\Delta\varphi_0 = \frac{\pi}{18}$, $t = 3$ (on the left) and the increments $\Delta\varphi_0 = \frac{\pi}{8}$, $t = 10$ (on the right) in the background strong (critical) winds $\tilde{W}$ (blue) and W (grey).

Lastly, let us also add that dating back to the formal genesis of the navigation problem in the Hamiltonian formalism, one could investigate the example under consideration with the use of the original navigation formula of E. Zermelo [25, 24, 10] combined with the results of A. De Mira Fernandes [12] since we chose here the Euclidean background. In particular case, i.e., $\mathbb{R}^2$ the problem was contemplated in [18]. Additionally, the equations of the limit curves which determine the planar area of the available points of arrivals as outlined on the right-hand side graph in Figure 4 one also might obtain. In this regards, for comparison to the initial research on the navigation problem in the classic calculus of variations and more details see § 282 - 287 in [10].

4 Discussion and concluding remarks

In our study we assumed that the norm $|u(x)|_h$ of a ship's velocity u, relative to the surrounding Riemannian sea (M, h) and being a spatial function of x is not necessarily constant, in particular unit, so unlike the standard concept and can be a priori fixed. In this sense we considered the generalization of the Zermelo navigation with the presence of a strong (critical) wind in a purely geometric form. Therefore, we aimed to be in line with the approach

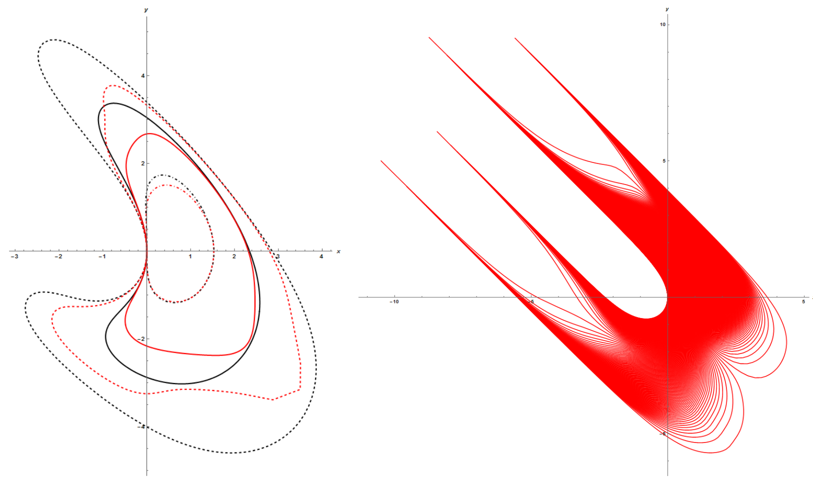

Figure 4: On the left the isochrones of the Kropina $\tilde{F}$- (red) and F-geodesics (black) starting from the origin, with $t = 1$ (dot-dashed), $t = 2$ (solid), $t = 3$ (dashed). On the right the Kropina $\tilde{F}$-geodesics starting from the origin, with the increments $\Delta\varphi_0 = \frac{\pi}{720}$, $t = 500$, outlining the area of available points of arrivals in $\mathbb{E}^2$.

to the problem presented in other contributions cited in the introduction and also referred to our previous study for the case of Randers metric, that is, the generalization of the navigation problem under a weak wind $|W_R|_h < |u|_h$. In a starting point we consider varying in magnitude speed $|u(x)|_h$ as predetermined control which complements standard navigation data $(h, \tilde{W})$. Having combined and compared our investigation to the referred results presented in [23] on Kropina metrics as well as making use of the original formulation of the Zermelo navigation with $h(u, u) = 1$, we can state that the difference in both approaches corresponds to the points of view at the problem and the solutions are connected in a simple manner. In what follows, we discuss some details and collect the findings.

In fact, the introduced new data is strongly limited by the main assumption on the norms' equality, i.e., $|u|_h = |\tilde{W}|_h$ which determines the case. One may imagine that in the scenario under consideration there are the "speed zones" referring to the ship's speed through the water or, in other words, the "speed limits" which cover the whole Riemannian sea (M, h). Therefore, captain's duty is to take them into account when preparing the passage plan for a time-efficient voyage by continuous adjusting a ship's engine on the entire route. From another point of view a ship's engine telegraph-based plan is executed and the wind force is to be adapted to the fixed ship's passage plan such that it is time-efficient. Though the latter scenario is far away from the real marine or air navigation, there are applied optimal control problems when just acting perturbation is fully controllable.

Proposition 2.2 establishes the direct relation between the Kropina geodesics and the time-minimal paths as the solutions to the navigation problem introducing the space-dependent function $|u(x)|_h$. We see that under the action of a wind $\tilde{W}$ the time-efficient travel path, so the solution to the generalized Zermelo problem, is no longer a background Riemannian h-geodesic, but a geodesic of the $\tilde{U}$-Kropina metric $\tilde{F}$. For comparison, let us reflect for a moment on the generalized Randers case (cf. [13, 15]), where one might ask if decreasing a ship's speed $|u|_h$ under fixed weak wind field W_R causes the same effect on the time-minimal path as increasing the wind force with $h(u,u)=1$ and holding the same relation $\frac{|u|_h}{|W_R|_h}$. Since $0 \leq |W_R|_h < |u|_h < 1$, the decrease of a ship's relative speed introduces a relatively larger effective wind $\tilde{W}^i_R > W^i_R$. From this point of view the formula for Randers metric in the presented generalization is then given as in the original setting [5], however with W^i_R replaced by a rescaled wind $\tilde{W}^i_R = \frac{1}{|u(x)|_h} W^i_R$. Now, in the presence of stronger perturbation we followed our approach presented in the Randers case what increases the variety of the scenarios and the solutions influenced by the new spatial function $|u(x)|_h$.

Remark that according to Corollary 2.3 the corresponding travel times are greater in comparison to the standard expression of the problem. Actually, having admitted a priori the space-dependence of $|u(x)|_h$ our presentation makes a difference in the genesis in comparison to [23]. Note that changing $|u(x)|_h$ in the generalized Randers case does not entail the modification of navigation data (h, W_R). Under a critical perturbation one who sets $|u(x)|_h$ initially may state that it affects the scalar correspondance as the strict condition $|u(x)|_h = |\tilde{W}(x)|_h$ is in force. Let us also remark that unlike the Randers case, where the entire space can be covered with the time-minimal paths, now not all the destinations are available any more due to the fact that the wind became stronger. In further research one could obtain the general equations or the conditions to be fulfilled for the limit curves which determine the subspace of M including the flows of Kropina geodesics for given generalized navigation data $(h, |u|_h, \tilde{W})$. Therefore, the maps on M including the areas of existing connections traveled in the presence of a strong (critical) wind could complement the findings.

The solutions to the Zermelo problem are represented in the original and the generalized formulation by the same paths up to scaling if $|u|_h = const.$, that is F- and $\tilde{F}$-geodesics trace the same curves. Such a case corresponds to a pair of conformal homothetic Finsler metrics, that is a special case of weakly conformally equivalent Finsler metrics considered in [20]. The travel times differ then due to the influence of variable $|u(x)|_h$ or, equivalently, $|W(x)|_h$. By Corollary 2.3 together with Corollary 2.1, the consequence is the fact that applying any $|u|_h \neq 1$ the passage time increases in comparison to the original expression which determines the conformal solution of absolutely minimal time. Furthemore, the bijection is established between Kropina spaces represented by pairs $(\tilde{\alpha}, \tilde{\beta})$ and (h, W) or triples $(h, |u|_h, \tilde{W})$, where $\tilde{W}^i = |u|_h W^i$. Therefore, the generalization with a spatial function $|u(x)|_h$ in the presence of a strong wind refers to the standard problem with normalized wind, i.e. $W = \frac{\tilde{W}}{|\tilde{W}|_h}$. This conclusion is in line with the theory on globally defined U-Kropina metrics (cf. [23]), where it follows that any Riemannian manifold (M, h) that admits a globally defined nowhere vanishing vector field W can be endowed with a globally defined U-Kropina metric. In order to see this, it is remarked that for a Riemannian metric h and a vector field W on M without zeros, it is enough to normalize $\tilde{W}$. Then one can construct a U-Kropina metric using h and W. In fact, the correlated studies coming from the slightly different starting points of view at the navigation problem meet. This is caused directly by the main assumption on the norms' equality which determines a very special case of the Zermelo navigation problem treated in the paper.

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References

[1] N. Aldea. *Complex Finsler spaces with Kropina metric.* Bull. Transilvania Univ. Braşov (Proc. Conference RIGA 2007) **14** (49s.) (2007) 1–10.

[2] N. Aldea, P. Kopacz. *Generalized Zermelo navigation on Hermitian manifolds under mild wind.* Diff. Geom. Appl. (2017), in press.

[3] N. Aldea, P. Kopacz. *Generalized Zermelo navigation on Hermitian manifolds with a critical wind.* Results Math. (2017) 72(4): 2165-2180, doi:10.1007/s00025-017-0757-6.

[4] K. J. Arrow. *On the use of winds in flight planning.* J. Meteor. **6** (1949) 150-159.

[5] D. Bao, C. Robles, and Z. Shen. *Zermelo navigation on Riemannian manifolds.* J. Diff. Geom. **66**(3) (2004) 377–435.

[6] D. C. Brody, G. W. Gibbons, and D. M. Meier. *Time-optimal navigation through quantum wind.* New J. Phys. **17** (033048) (2015).

[7] D. C. Brody, G. W. Gibbons, and D. M. Meier. *A Riemannian approach to Randers geodesics.* J. Geom. Phys. **106** (2016) 98–101.

[8] D. C. Brody and D. M. Meier. *Solution to the quantum Zermelo navigation problem.* Phys. Rev. Lett. **114** (100502) (2015).

[9] E. Caponio, M. A. Javaloyes, and M. Sánchez. *Wind Finslerian structures: from Zermelo's navigation to the causality of spacetimes.* arXiv:1407.5494, 2015.

[10] C. Carathéodory. *Calculus of Variations and Partial Differential Equations of the First Order*. AMS, Chelsea Publishing, 1935 (reprint 2008).

[11] S.-S. Chern and Z. Shen. *Riemann-Finsler geometry.* Nankai Tracts in Mathematics. World Scientific, River Edge (N.J.), London, Singapore, 2005.

[12] A. De Mira Fernandes. *Sul problema brachistocrono di Zermelo.* Rendiconti della R. Acc. dei Lincei **XV** (1932) 47–52.

[13] C. A. R. Herdeiro. *Mira Fernandes and a generalised Zermelo problem: purely geometric formulations.* Bol. Soc. Port. Mat., Special Issue. Proc. of "Mira Fernandes and his age - An historical conference in honour of Aureliano de Mira Fernandes (1884-1958)"', Instituto Superior Técnico, Lisboa, 179-191, June 2009 (Eds. L. Saraiva and J. T. Pinto), Lisboa, 1–13, 2010.

[14] P. Kopacz. *Application of planar Randers geodesics with river-type perturbation in search models.* Appl. Math. Model. **49** (2017) 531–553.

[15] P. Kopacz. *On generalization of Zermelo navigation problem on Riemannian manifolds.* arXiv:1604.06487 [math.DG], 2016.

[16] P. Kopacz. *A note on time-optimal paths on perturbed spheroid.* arXiv:1602.00167 [math.DG], 2016.

[17] T. Levi-Civita. *Über Zermelo's Luftfahrtproblem.* ZAMM-Z. Angew. Math. Me. **11**(4) (1931) 314–322.

[18] B. Manià. *Sopra un problema di navigazione di Zermelo.* Math. Ann. **133** (3) (1937) 584–599.

[19] V. S. Matveev, H.-B. Rademacher, M. Troyanov, and A. Zeghib. *Finsler conformal Lichnerowicz-Obata conjecture*. Ann. I. Fourier **59** (3) (2009) 937–949.

[20] M. Rafie-Rad. *Weakly conformal Finsler geometry*. Math. Nachr. **287** (14-15) (2014) 1745–1755.

[21] B. Russell and S. Stepney. *Zermelo navigation and a speed limit to quantum information processing*. Phys. Rev. A **90** (012303) (2014).

[22] B. Russell and S. Stepney. *Zermelo navigation in the quantum brachistochrone*. J. Phys. A - Math. Theor. **48** (115303) (2015).

[23] R. Yoshikawa and S. V. Sabau. *Kropina metrics and Zermelo navigation on Riemannian manifolds*. Geom. Dedicata **171** (1) (2013) 119–148.

[24] E. Zermelo. *Über die Navigation in der Luft als Problem der Variationsrechnung*. Jahresber. Deutsch. Math.-Verein. **89** (1930) 44–48.

[25] E. Zermelo. *Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung*. ZAMM-Z. Angew. Math. Me. **11** (1931) 114–124.

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